Aircraft Performance Optimization

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Using the calculus of variations, the solutions to various fixed end point flight-path optimization problems are developed. These include the minimum fuel-fixed range problem, the minimum time-fixed range problem, and the minimum fuel-fixed range-fixed time problem. Altitude profiles and throttle control laws are presented. A variety of aircraft mathematical models is initially examined, and the existence of a classically optimal controller is verified for a simple model. For this model, the first integral condition is used to eliminate the requirement of integrating the Euler Lagrange adjoint differential equations. The resulting computational algorithms are attractive for both laboratory investigations and airborne implementations.

Nomenclature

D = drag
E = energy
G = performance criterion
g = gravity
H = Hamiltonian
h = altitude
J = performance index
K = induced drag coefficient
L = lift force
M = mass
Q = dynamic pressure
q = pitch rate
S = aerodynamic reference area
T = thrust
t = time
u = control variable
V = velocity
X = state variable
x = distance
v = velocity set
λ = Euler Lagrange variable
π = throttle setting
α = angle of attack
γ = flight-path angle
σ = specific fuel consumption
θ = pitch angle
δ = elevator deflection
$C_{m\alpha}$, $C_{m\delta}$ = aerodynamic pitching moment derivatives
$C_{z\alpha}$, $C_{z\delta}$ = aerodynamic lift coefficient derivatives
C_{d_0} = parasite drag coefficient
I_y = moment of inertia
V_s = switching velocity
δ_s = stick deflection

Subscripts

$(\ )_c$ = cruise condition
$(\ )_f$ = values at final time
$(\ )_0$ = values at initial time

Introduction

AIRCRAFT performance optimization has been the subject of many investigations. However, most of the results have been developed only for segments of the trajectory. Rutowski[1] considered the minimum time and minimum fuel climb from a graphical viewpoint. The time-to-climb problem was subsequently treated by Lush,[2] Garfinkle,[3] Miele,[4] and Cicala and Miele.[5] Kelly[6] and Bryson[7] considered climb problems using digital gradient techniques. Most recently Bryson, et al.[8] have developed a simple computational means for producing the approximate Rutowski minimum fuel and minimum time paths. The minimum fuel-cruise problem has also been treated extensively (e.g., the well-known Breguet formulas). Maximum range glide solutions also form a part of the classical solutions.[4]

Only limited information is available on optimization of trajectories from initial point to final point (climb-cruise-descent). Stein, et al.[9] investigated the supersonic optimum trajectory problem using steepest descent methods. Solutions were obtained from lengthy and complicated digital programs.

Bryson, et al.[8] obtained very simple mission trajectory solutions by applying optimization principles to the energy state equations. This procedure yields some curious results; namely, that partial throttle control cannot occur even for minimum fuel cruise-type trajectories. Schultz, et al.[10] have shown that for a different set of equations, partial throttle can occur.

Zagalsky, et al.[11] have shown that for the energy state equation, a relaxed control can produce performance better than that of the optimal solutions obtained from the energy state equation. The reason was that, for the example considered, the velocity set was not convex and the maximal principle did not apply. In this paper these inconsistencies are resolved by examining various forms of the equation of motion; then, optimal solutions are obtained using a set of simple equations which generally agrees with the more exact equation.

Conclusions

The maximum principle does not always result in maximum aircraft performance because with some sets of aircraft motion equations the velocity set is not convex and a relaxed controller is possible. However, simple sets of motion equations do exist which can be used with the maximal principle and have the same general properties of the more exact equation.

Using these motion equations, a simple set of solution equations can be obtained. The Euler-Lagrange differential equations and the maximum principle reduce to two differential equations and a maximum principle where all of the Euler-Lagrange variables are eliminated from the solution equations.

These solution equations can be used in three ways: 1) during the aircraft design, the effects of aircraft parameters on performance can be quickly and systematically determined; 2) aircraft performance curves for pilot's handbooks can be quickly and efficiently generated; and 3) the solutions can be generated onboard, to provide the pilot with real-time display projections of trajectory information.

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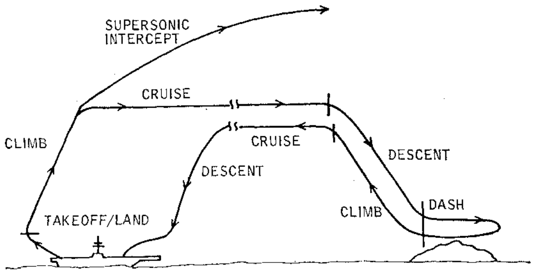

Fig. 1 Military aircraft Mission

Problem statement

The problem addressed in this investigation is to find simple methods of optimizing aircraft performance for an entire mission. A typical mission for a military aircraft is an attack mission illustrated in Fig. 1. To perform a successful mission, the aircraft may have to be at a specified place at a specified time, and usually some performance criterion such as minimum fuel or minimum time must be optimized, within some physical constraints on the system.

This type of performance problem can be stated mathematically as follows:

Find the control variable (u) that maximizes a performance index

$$J = \int_0^{t'} G(\bar{x}, \bar{u}, t)dt \tag{1}$$

where $\bar{x}$ and $\bar{u}$ must also satisfy the differential equations

$$\dot{\bar{x}} = \bar{f}(\bar{x},\bar{u},t) \tag{2}$$

and $\bar{x}$ may have initial and terminal conditions of the form

$$\bar{x}(t_0) = \bar{x}_0 \qquad \bar{x}(t_f) = \bar{x}_f \tag{3}$$

and inflight constraints of the form may exist

$$\bar{u} \in \bar{V}_c \tag{4}$$

Aircraft performance problems which can be formulated according to Eqs. (1–4) can be solved by the Pontryagin Maximum Principle.[12]

The solution to the stated optimization problem is:

The control variables are determined from

$$\min_{\bar{u}\in\bar{V}_c} H \qquad H = G + \bar{\lambda}^T\bar{F} \tag{5}$$

where the $\bar{\lambda}$'s and $\bar{x}$'s are determined from:

$$\begin{aligned} &\dot{\bar{x}} = \bar{F}(\bar{x},\bar{u},t) && \bar{x}(t_0) = \bar{x}_0 \\ &\dot{\bar{\lambda}} = -\partial H/\partial\bar{x} && \bar{x}_j(t_f) = \bar{x}_{jf} \qquad j = 1 \ldots k \\ & && \qquad\qquad\qquad k < m \end{aligned} \tag{6}$$

The boundary conditions are determined from the transversality conditions which are

$$H\, dt_f - \bar{\lambda}^T\, d\bar{x}_f = 0 \tag{7}$$

the solution must also satisfy the first integral

$$dH/dt + \partial H/\partial t = 0 \tag{8}$$

To ensure existence of the optimal solution, the velocity set[13] ν, where ν is:

$$\nu = \{\bar{F}, G \mid \bar{u} \in \bar{V}_c\}$$

must be convex.

Aircraft Models

To address the solution, a mathematical description of the aircraft and its motion is required. Equations of varying degrees of complexity can be used to describe aircraft motion. They are, in order of decreasing complexity: 1) Flight-path equations—moment equations. 2) Point mass equations. 3) Point mass equations—small angle of attack. 4) Point mass equations—vertical force equilibrium ($L = W$) = small flight-path angle ($\gamma = 0$). 5) Energy state approximation.

These equations are presented in Appendix A. There, the maximal principle is applied to the equations and the properties of the control variables are investigated. The properties of the velocity set are also investigated. The results of this analysis are presented in Table 1.

This table shows that a partial throttle control is possible for equation sets 1, 3, and 4. Partial throttle is ruled out by the maximum principle in cases 2 and 5. However, when the velocity set is not convex, as occurs in cases 2, 3, and 5, a relaxed controller may exist and the maximal principle may not apply.

The different conclusions are apparently due to the approximations which are made. In equation set 1, the angle of attack is a state variable and is obtained from a differential relationship from the control variable surface deflection while in equation set 2, α is a control variable and can be instantaneously attained. Thus, the relaxed controller probably appears in set 2 because the rate constraints on angle of attack are removed.

This conclusion is consistent with the observation that a relaxed controller can be eliminated by putting rate constraints on the control variable.

A similar relationship occurs between equation sets 4 and 5. In set 5, velocity is allowed to be attained instantaneously. Equation set 4 puts a rate constraint on the velocity and removes the possibility of a relaxed controller.

Equation set 1 is the most exact representation of aircraft motion and the control variables correspond most closely to physical controls. Thus, if an approximate set of equations is used with the maximal principle to determine optimum performance, the general properties of the solution should agree with those of equation set 1.

Table 1 Properties of equation sets

	Equation sets 1	2	3	4	5
Control variables	π, δ_s	π, α	π, α	π, γ	π, V
Control variable properties from maximum principle	$\pi = \begin{cases}\text{max}\\\text{min}\\\text{partial}\end{cases}$	$\pi = \begin{cases}\text{max}\\\text{min}\end{cases}$	$\pi = \begin{cases}\text{max}\\\text{min}\\\text{partial}\end{cases}$	$\pi = \begin{cases}\text{max}\\\text{min}\\\text{partial}\end{cases}$	$\pi = \begin{cases}\text{max}\\\text{min}\end{cases}$
	$\delta_s = \begin{cases}\text{max}\\\text{min}\\\text{partial}\end{cases}$	$\alpha = \begin{cases}\text{max}\\\text{min}\\\text{partial}\end{cases}$	$\alpha = \begin{cases}\text{max}\\\text{min}\\\text{partial}\end{cases}$	$\gamma = \begin{cases}\text{max}\\\text{min}\\\text{partial}\end{cases}$	$V = \begin{cases}\text{max}\\\text{min}\\\text{partial}\end{cases}$
Velocity set	Convex	Not convex	Not convex	Convex	Not convex

Equation set 4 is the simplest set that has the desired properties. Thus it will be used in the investigation of some performance problems.

Optimal Performance Solutions

Minimum Fuel—Fixed Range

In this problem the minimum fuel for a climb-cruise-descent trajectory from an initial altitude, range, and velocity to a terminal range, altitude, and velocity is to be found. Terminal time is not specified.

The performance criterion and the boundary conditions are:

$$\begin{array}{ll} G=\sigma T & E(t_0)=E_0 \\ E(t_f)=E_f & x(t_0)=x_0 \\ x(t_f)=x_f & h(t_0)=h_0 \\ h(t_f)=h_f & \end{array} \tag{10}$$

Using the equations of motion and the maximal principle, the equations to be solved are:

$$\begin{array}{l} \dot{E}=(T-D)V/M, \qquad \dot{h}=V\gamma, \qquad \dot{x}=V \\ \min\limits_{\pi,\gamma}\,[G+\lambda_1(T-D)V/M+\lambda_2 V\gamma+\lambda_3 V]_{E,h} \\ \dot{\lambda}_1=-(\partial H/\partial E)_h \qquad \dot{\lambda}_2=-(\partial H/\partial h)_E \\ \dot{\lambda}_3=-(\partial H/\partial x) \end{array} \tag{11}$$

The transversality conditions and the first integral are:

$$H=\sigma T+\lambda_1(T-D)V/M+\lambda_2 V\gamma+\lambda_3 V=\text{const}=H_0$$

$$Hdt-\lambda_1 dE-\lambda_2 dh-\lambda_3 dx\,\big|_{ti}^{tf}=0 \tag{12}$$

Performing the maximization, the possible controls are:

$$\pi=\begin{cases}\pi_{\max} & \text{if} \quad \sigma+\lambda_1 V/M<0\\ \pi_{\min} & \text{if} \quad \sigma+\lambda_1 V/M>0\\ \text{partial} & \text{if} \quad \sigma+\lambda_1 V/M=0\end{cases}$$

$$\gamma=\begin{cases}\gamma_{\max} & \text{if} \quad \lambda_2<0\\ \gamma_{\min} & \text{if} \quad \lambda_2>0\\ \text{partial} & \text{if} \quad \lambda_2=0\end{cases} \tag{13}$$

For portions of the trajectory, we assume that $\gamma_{\max}>\gamma>\gamma_{\min}$, then $\lambda_2=0$ and $\dot{\lambda}_2=0$, which from Eq. (11) implies

$$(\partial/H)/\partial h)_{E,\pi}=0 \tag{14}$$

The three throttle conditions can now be considered. For $\pi=\pi_{\max}$, Eq. (14) is

$$\partial/\partial h[\sigma T+\lambda_1(T-D)V/M+\lambda_3 V]_E=0 \tag{15}$$

A form equivalent to Eq. (15) with λ_1 eliminated is:

$$\frac{\partial}{\partial h}\left[\frac{(T-D)V/M}{\sigma T+\lambda_3 V}\right]_E=0 \tag{16}$$

This can be verified by

$$\frac{\partial}{\partial h}\left(\frac{A}{B}\right)=\frac{1}{B}\frac{\partial A}{\partial h}-\frac{A}{B^2}\frac{\partial B}{\partial h}=-\frac{A}{B^2}\left(-\frac{B}{A}\frac{\partial A}{\partial h}+\frac{\partial B}{\partial h}\right) \tag{17}$$

and using $H=0$ from Eq. (12):

$$\lambda_1=-\frac{\sigma T+\lambda_3 V}{(T-D)V/M}=-\frac{B}{A} \tag{18}$$

Note that the constant value of λ_3 is not yet known. If $\pi=_{\min}=0$, then Eq. (16) becomes

$$(\partial/\partial h)(D/M)_E=0 \tag{19}$$

Note that the maximum total mission glide range relation given in Eq. (19) would be different if the minimum fuel flow were nonzero.

For the partial throttle condition

$$\sigma+\lambda_1 V/M=0 \tag{20}$$

Take derivative

$$\dot{\lambda}_1 V/M+(\partial\sigma/\partial V+\lambda_1/M)\dot{V}=0 \tag{21}$$

Assume that $T=D$, then $\dot{\lambda}_1=0$ or also

$$(\partial/\partial E)[\sigma T+\lambda_1(T-D)V/M+\lambda_3 V]=0 \tag{22}$$

Expanding Eq. (15)

$$(\partial/\partial h)(H)_E=\lambda_3(\partial V/\partial h)_E+(\partial\sigma D/\partial h)_E=0 \tag{23}$$

using $H=0$

$$\lambda_3=-\sigma D/V \tag{24}$$

then Eq. (15) can be written as

$$\partial/\partial h(\sigma D/V)_E=0 \tag{25}$$

In the same manner, Eq. (22) reduces to

$$(\partial/\partial E)(\sigma D/V)_h=0 \tag{26}$$

or equivalently

$$(\partial/\partial V)(\sigma D/V)_h=0 \tag{27}$$

Thus, the cruise point is given by

$$\begin{array}{l}(\partial/\partial h)(\sigma D/V)_V=0\\ (\partial/\partial V)(\sigma D/V)_h=0\end{array} \tag{28}$$

or if the partials exist by

$$\min_{h,V}\left(\frac{\sigma D}{V}\right) \tag{29}$$

After the cruise point is found from Eq. (24), λ_3 can be determined

$$\lambda_3=-\sigma_c D_c/V_c \tag{30}$$

where σ_c, D_c and V_c are the cruise condition.

Equation (16) then becomes

$$\frac{\partial}{\partial h}\left[\frac{(T-D)V/M}{\sigma T-\sigma_c D_c/V_c}\right]_E=0 \tag{31}$$

If a short range is to be traveled, a climb to the cruise point may not be required. Then the switch from full throttle to minimum throttle is determined from

$$\sigma+\lambda_1 V/M\leq 0 \tag{32}$$

λ_1 can be eliminated in terms of λ_3 by using Eq. (12)

$$\sigma-\frac{(\sigma T+\lambda_3 V)V/M}{(T-D)V/M}\leq 0 \tag{33}$$

or the switch from full throttle to minimum throttle occurs, where

$$-\sigma D/V-\lambda_3\leq 0 \tag{34}$$

or when

$$-\sigma D/V\leq\lambda_3 \tag{35}$$

Thus, if λ_3 is chosen to be greater than the minimum value of $-(\sigma D/V)$, the trajectory segment will consist only of $\pi=\pi_{\max}$ and $\pi=\pi_{\min}$ segments. The value chosen for λ_3 will determine when the throttle is switched and consequently the range traveled. The switching curves, which are curves of constant $\sigma D/V$ are typically as shown in Fig. 2.

The curves developed thus far are for the singular condition; i.e., $(\lambda_2=0)$. These curves, however, are not connected. They must be connected by an arc along which

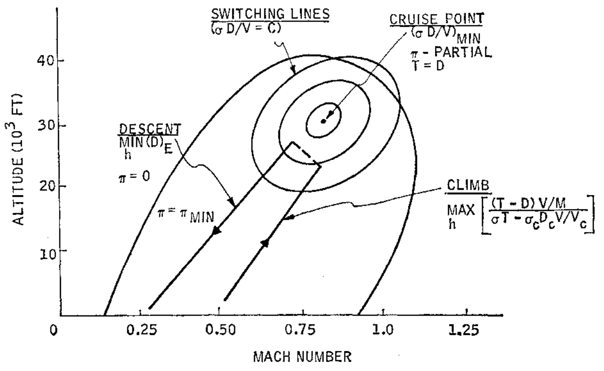

Fig. 2 Minimum fuel—fixed range solution.

$\gamma = \gamma_{\max}$. These arcs are not determined and it is assumed that they occur in zero time and zero range.

Summarizing, the trajectory is determined from

$$\dot{E} = (T - D)V/M \qquad E(t_0) = E_0$$
$$\dot{x} = V \qquad x(t_0) = x_0 \tag{36}$$

The control and other variables are determined from

Climb

$$\max_h \left[\frac{(T-D)V/M}{\sigma T - \sigma_c D_c V/V_c}\right]_E \tag{37}$$
$$\pi = \pi_{\max}$$

Cruise

$$\min_{h,V}(\sigma D/V)$$
$$T = D$$

Descent

$$\min_h (D/M)_E$$
$$\pi = 0$$

The climbing portion of the curve occurs at maximum throttle and its altitude-velocity relationship is determined by the maximization operation. The function to be maximized is similar to the function for the Rutowski minimum fuel climb with no specified range which is $[(T - D)V/M)]/\sigma T$. The additional term in the denominator of the function is caused by the range condition. The cruise solution occurs at partial throttle and the cruise altitude and velocity are found by finding the global minimum of $\sigma D/V$ inside the $T - D = 0$ envelope. If the minimum of $\sigma D/V$ *w.r.t.* V, h does not occur inside the $T - D = 0$ boundary, then the cruise occurs at maximum throttle on the $T - D = 0$ curve and h is determined from min $\sigma D/V$ with respect to h. Descent occurs at zero throttle and occurs along the minimum drag curve or along zero altitude.

The cruise solution does not necessarily need to occur. The trajectory can consist of only a maximum throttle climb and a minimum fuel descent. The h-V profile for this case is determined from $\max[(T - D)V/M]/(\sigma T - \lambda_3 V)$, where λ_3 is a constant such that $\lambda_3 > -\min(\sigma D/V)$. The exact value is picked to reach the end condition on range. The switch to minimum throttle is determined when $-\sigma D/V - \lambda_3 = 0$.

The solution for the case including the cruise portion can be computed on a digital computer by first determining the cruise altitude and velocity from Eq. (37). Equations (36) are then integrated while simultaneously solving Eq. (37) for the climb, altitude, and throttle. When the cruise altitude has been reached, the cruise throttle setting, altitude and velocity are determined from Eq. (37). The feasibility of using Eqs. (36) and (37) for airborne energy management is investigated in Ref. 14.

Minimum Time Intercept

Consider the problem of traveling at specified range in minimum time starting at a given energy level and ending at a given energy level. For this case

$$G = 1 \qquad E(t_0) = E_0$$
$$E(t_f) = E_f \qquad x(t_0) = x_0 \tag{38}$$
$$x(t_f) = x_f$$

The Hamiltonian is

$$H = 1 + \lambda_1(T - D)V/M + \lambda_2 V\gamma + \lambda_3 V \tag{39}$$

The traversality condition gives these additional boundary conditions:

$$H = 0, \; \lambda_3 = \text{const} \tag{40}$$

The solution can be obtained in a manner similar to that in the previous subsection.

During climb ($\pi = \pi_{\max}$) the altitude profile is determined from

$$\frac{\partial}{\partial h}\left[\frac{(T-D)V/M}{1 + \lambda_3 V}\right]_E = 0 \tag{41}$$

Note λ_3 is not yet known.

If the range to be traveled is short, the switch point from maximum throttle to minimum throttle is determined from

$$\pi = \pi_{\max} \text{ if } \lambda_1 V/M \le 0, \qquad \pi = 0 \text{ if } \lambda_1 V/M \ge 0 \tag{42}$$

Eliminating λ_1 in terms of λ_3, which is a constant, the switching conditions are

$$-\lambda_3 V + 1/(T - D) \le 0, \qquad 1 + \lambda_3 V \ge 0 \tag{43}$$

Defining $1/\lambda_3 = -V_S$, the switching areas are shown in Fig. 3.

If λ_3 is chosen to be $\lambda_3 = -1/V_{\max}$, then a cruise can occur at the flight envelope.

For descent

$$\partial/\partial h[1 - \lambda_1 DV/M + \lambda_3 V]_E = 0 \tag{44}$$

Eliminating λ_1 from using $H = 0$

$$\frac{\partial}{\partial h}\left(\frac{DV}{1 - V/V_S}\right)_E = 0 \tag{45}$$

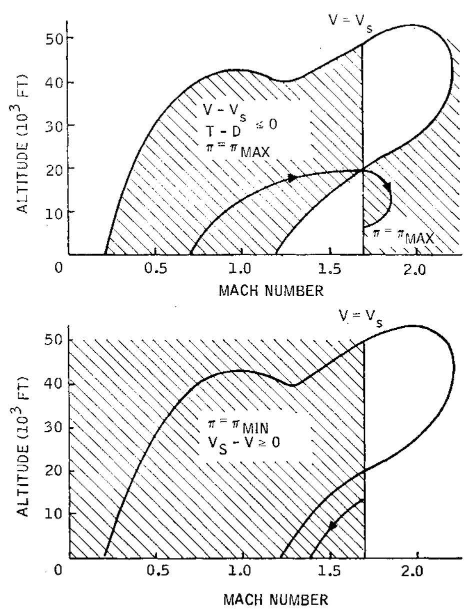

Fig. 3 Minimum time intercept switching lines.

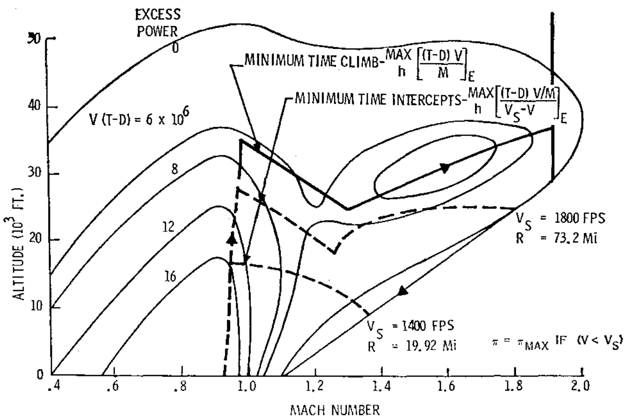

Fig. 4 Minimum time intercept solutions.

This derivative does not exist, but does indicate that the optimum solution occurs at

$$\max_h \left[\frac{DV}{1 - V/V_{\max}}\right]_E \tag{46}$$

Expression (46) does not have a maximum inside the placard limit, but its maximum value occurs on the placard limit. Note that this condition seems physically plausible because the rate of energy loss at zero throttle is

$$dE/dt = -DV/M \tag{47}$$

Thus to descend at the fastest possible rate while gaining range, (DV) should be maximized. Summarizing the solution equations are:

$$\dot{E} = (T - D)V/M \qquad \dot{x} = V \tag{48}$$

The controls are determined from :
Climb

$$\max_h \left[\frac{(T - D)V/M}{1 - V/V_S}\right] \tag{49}$$

$$\pi = \pi_{\max} \text{ (afterburner)}$$

Descent

Placard limit

$$\pi = 0$$

The solution for a minimum time intercept for an F-4-type aircraft is shown in Fig. 4, on an altitude Mach number plot.

The climb occurs at max throttle. The altitude-velocity profile along the optimum climb curve is determined by maximizing the function $[(T - D)V/M]/(1 - V/V_S)$. The function is similar to the function which is maximized for a minimum time climb to an energy level which is $(T - D)V/M$. The term $1 - V/V_S$ is due to the specified terminal range condition. The term causes the ascent trajectory to be shifted toward the higher velocity. It appears that emphasis is placed on gaining range by traveling at higher velocities, as well as gaining energy. Descent occurs at zero throttle. The descent altitude velocity profile is the placard limit. For longer ranges, a climb all the way up to $V_{\max}$ and a cruise there will occur.

On a digital computer, the solution must be obtained by iteration. V_S is first chosen, then the throttle is set to full afterburner. Equations (48) are integrated and the climb h and V are found from Eq. (49).

When V_S is reached, throttle is set to idle and the descent h and V are determined from Eq. (49). When the terminal energy is reached, the integration is stopped. If the terminal range is not reached, a new value of V_S is chosen.

Minimum Fuel-Fixed, Range-Fixed Time

This problem involves finding the minimum fuel trajectory from an initial range, energy, altitude, velocity, to a terminal altitude, range, velocity, with a specified time of arrival.

The boundary conditions are:

$$E(t_0) = E_0 \qquad E(t_f) = E_f \tag{50}$$
$$x(t_0) = x_0 \qquad x(t_f) = x_f \qquad t_f = t_F$$

The transversality conditions for this problem are:

$$H(t_f)dt_f - dE(t_f) - \lambda_2 dh(t_f) - \lambda_3 dx(t_f) = 0 \tag{51}$$

Thus the additional boundary conditions are:

$$H(t_f) = C_t \qquad \lambda_3(t_f) = \text{const} \tag{52}$$

The solution can be obtained by using the same procedure as in the minimum fuel-fixed range problem.

The solution equations for the trajectory are:

$$\dot{E} = (T - D)V/M \qquad \dot{x} = V \tag{53}$$

The controls and other variables are determined from:
Cruise

$$\min_{h,V} (\sigma D + \lambda_3 V)$$
$$T = D$$

Climb

$$\max_h \left[\frac{(T - D)V/M}{\sigma T - \sigma_c D_c + \lambda_3(V - V_c)}\right]_E \tag{54}$$

$$\pi = \pi_{\max}$$

Descent

$$\min_h \left[\frac{DV/M}{-\sigma_c D_c + \lambda_3(V - V_c)}\right]_E$$

$$\pi = 0$$

To solve the problem λ_3 and the cruise time (t_c) must be adjusted until both the terminal range and terminal time are met. A special case occurs when λ_3 is chosen to be zero. This corresponds to a minimal fuel, fixed-time, fixed-energy problem with no terminal range condition. However, if the $\lambda_3 = 0$ solution provides a range longer than the range required, then the terminal range can be hit by expending range in a circle.

The solution for $\lambda_3 = 0$ is shown in Fig. 5 on an h, M grid.

The climb occurs at maximum throttle along an altitude velocity profile determined by max $[(T - D)V/M]/(\sigma T - \sigma_c D_c)$. This function is similar to the function which is minimized for a minimum fuel climb, cruise, descent to a specified range but the second term in the denominator is different. This difference is caused by the terminal time constraint.

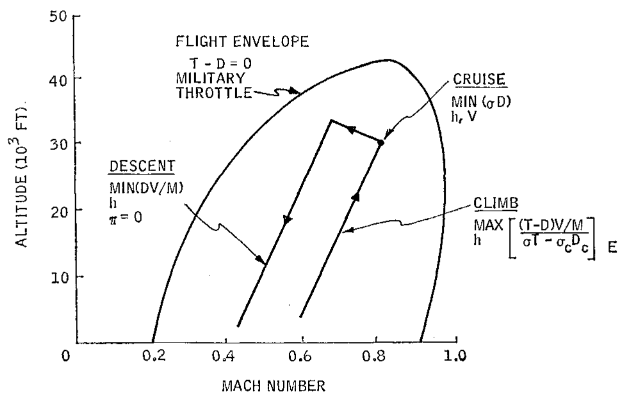

Fig. 5 Minimum fuel, fixed range, fixed time solution.

Cruise occurs at partial throttle and at the global minimum of (σD) with respect to h, and V. This point is different from the previous case where cruise occurred at minimum of $\sigma D/V$ with respect to h, and V. This cruise solution produces the minimum fuel rate ($\dot{m} = \sigma D$) with respect to time. Descent occurs at zero throttle along a maximum time descent, $\min(DV/M)$, thus as much of the total specified time interval as possible is spent at a zero throttle descent.

The solution requires three switches in throttle. These switches must occur so that both the correct terminal range and the correct terminal time are reached. One method which can be used to adjust time and range is to perform a horizontal circular turn at small roll angles ($\cos\phi \cong 1$). This takes time but no range is traveled.

The time spent circling is

$$t_{\text{circle}} = t_f - (t_d + t_{\text{circle}} + t_{\text{climb}}) \tag{55}$$

The radius of the turning circle is

$$r_{\text{circle}} = V_c t_{\text{circle}}/2 \tag{56}$$

The circling is done at the cruise altitude and velocity. The feasibility of using these solution equations for shipborne air traffic control is investigated in Ref. 10.

Appendix A

Aircraft Dynamical Models

The equations for various aircraft models are presented in this appendix. The models are: 1) point mass—moment equations, 2) point mass equations, 3) point mass—small angle of attack, 4) point mass—vertical force equilibrium—small flight path angle, and 5) energy state equations.

First, the maximal principle is applied and the properties of the control variable solutions are investigated. Then the properties of the velocity set[13] are investigated. Existence of the optimal control solution obtained from the maximal principle is ensured if the velocity set is convex with respect to the control variables. However, if the velocity set is not convex, the optimal solution may still exist or a relaxed controller may be possible. In the latter case, a determination of the solution type requires further investigation of the particular properties of the velocity set.

1. *Flight Path Equations—Moment Equations*

The point mass flight-path and moment equations are:

$$\begin{gathered}\dot{q} = QS(C_{m\alpha}\alpha + C_{m\delta}\delta)/I_y, \qquad \dot{\theta} = q \\ \dot{E} = (T\cos\alpha - D)V/M, \quad \dot{\gamma} = (L + T\sin\alpha - W\cos\gamma)/MV \\ \dot{x} = V\cos\gamma, \qquad \dot{h} = V\sin\gamma, \qquad \dot{\delta} = 1/\tau(\delta_s - \delta)\end{gathered} \tag{A1}$$

where E is the energy and is defined as

$$E = V^2/2 + gh \tag{A2}$$

where the lift and drag are

$$\begin{aligned} L &= QS(C_{z\alpha}\alpha + C_{z\delta}\delta) \\ D &= QS(C_{do} + KC_L{}^2) \\ T &= T^*\ (h, V)\pi \end{aligned} \tag{A3}$$

The state variables in this set of equations are q, θ, E, γ, x h, and m. The control variables are π and δ_s.

For this set of equations, H is

$$\begin{aligned} H = {} & G + \lambda_1(M_\alpha\alpha + M_\delta\delta) + \lambda_2 q \\ & + \lambda_3(T\cos\alpha - D)V/M + \lambda_4(L + T\sin\alpha - W\cos\gamma)/MV \\ & + \lambda_5 V\sin\gamma + \lambda_6 V\cos\gamma + \lambda_7/\tau(\delta_s - \delta) \end{aligned} \tag{A4}$$

Similar expressions can be obtained for minimum time. For partial throttle and partial elevator to exist simultaneously, the following condition must occur:

$$\frac{\partial H}{\partial \pi} = 0, \ \frac{\partial H}{\partial \delta_s} = 0, \qquad \left[\frac{\partial^2 H}{\partial u^2}\right] \geq 0 \tag{A5}$$

$$\frac{\partial H}{\partial \pi} = \sigma + \lambda_3 \cos\alpha V/M + \lambda_4 \sin\alpha/MV = 0, \frac{\partial H}{\partial \delta_s} = \frac{1}{\tau}\lambda_7$$

$$\left[\frac{\partial^2 H}{\partial \pi \partial \delta_s}\right] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0 \tag{A6}$$

Thus, it is possible for partial throttle and partial elevator to exist simultaneously. However, for the maximal principle to apply the velocity set must be convex. For this set the velocity set is convex because the controls appear linearly and uncoupled in all expressions.

2. *Point Mass Equations*

The equations for a point-mass, three-degree-of-translational-freedom are:

$$\begin{aligned} \dot{E} &= (T\cos\alpha - D)V/M \\ \dot{\gamma} &= (L + T\sin\alpha - Mg\cos\gamma)/MV \\ \dot{h} &= V\sin\gamma \\ \dot{x} &= V\cos\gamma \end{aligned} \tag{A7}$$

The control variables are throttle angle (π) and angle of attack (α). The Hamiltonian is

$$\begin{aligned} H = {} & \sigma T + \lambda_1(T\cos\alpha - D)V/M \\ & + \lambda_2(L - T\sin\alpha - Mg\cos\gamma)/MV \\ & + \lambda_3 V\sin\gamma + \lambda_4 V\cos\gamma \end{aligned} \tag{A8}$$

For partial throttle to occur, conditions (A5) must be satisfied:

$$\partial H/\partial\pi = (\sigma + \lambda_1 \cos\alpha V/M + \lambda_2 \sin\alpha/MV)T^* = 0$$

$$\frac{\partial H}{\partial \alpha} = \lambda_1(1 - T\sin\alpha - \partial D/\partial\alpha)V/M + \lambda_2\left(\frac{\partial L}{\partial \alpha} + T\cos\alpha\right)/MV = 0 \tag{A9}$$

$$\left[\frac{\partial^2 H}{\partial u^2}\right] = -\left(\frac{\lambda_2 L_2}{T\sin\alpha + \partial D/\partial\alpha}\right)^2 (1 - \sin\alpha + 2K\alpha\cos\alpha)$$

The second partial is not positive semi-definite so partial throttle cannot occur, which disagrees with the conclusion reached for equation set 1. The velocity set shown in Fig. 6 is not convex. Furthermore, the intercept of the velocity set with the ($L + T\sin\alpha = W\cos\gamma$) plane in the area of a possible cruise point ($T = D$) shows that a relaxed controller is a definite possibility. Thus, the maximum principle probably does not apply for this set of equations, at least for the minimum fuel performance criterion. However, the curvature or concavity is small numerically and the percentage reduction in fuel flow with a relaxed controller is small.

3. *Point mass equations—small angle-of-attack approximation*

If the angle of attack is assumed to be small, the previous set of equations reduces to

$$\begin{gathered} \dot{E} = (T - D)V/M, \qquad \dot{\gamma} = (L - W\cos\gamma)/MV, \\ \dot{h} = V\sin\gamma, \qquad \dot{x} = V\cos\gamma, \qquad \dot{m} = \sigma T \end{gathered} \tag{A10}$$

Here H is

$$\begin{aligned} H = {} & \sigma T + \lambda_1(T - D)V/M + \lambda_2(L - W\cos\gamma)/MV \\ & + \lambda_3 V\sin\gamma + \lambda_4\ V\cos\gamma \end{aligned} \tag{A11}$$

H is linear in the control variables π and α, and the control variables are not coupled, thus the matrix of the second partial is identical to zero. The velocity set, shown in Fig. 7,

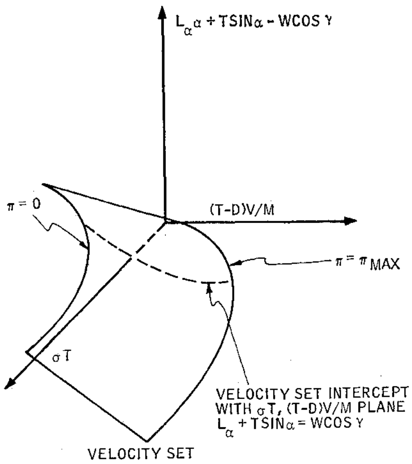

Fig. 6 Velocity set for equation set 2.

however, is not convex but its shape is such that a relaxed controller cannot reduce the fuel consumption.

4. *Point mass—equilibrium of vertical forces—small flight path angle*

The point mass equations can be simplified if vertical equilibrium of forces is assumed, i.e., $L = W$. This assumption reduces the number of equations by one. The equations are:

$$\dot{E} = (T - D)V/M, \qquad \dot{h} = V\gamma, \qquad \dot{x} = V \tag{A12}$$

For these equations, the control variables are π and γ. The Hamiltonian is

$$H = \sigma T + \lambda_1(T - D)V/M + \lambda_2\gamma + \lambda_3 V \tag{A13}$$

Again, checking the matrix of second partials

$$\frac{\partial H}{\partial \pi} = \sigma + \lambda_1 V/M, \ \frac{\partial H}{\partial \gamma} = \lambda_2, \ \left[\frac{\partial^2 H}{\partial u^2}\right] = [0] \tag{A14}$$

Thus, partial throttle control can exist which agrees with the conclusions reached with equations set 1.

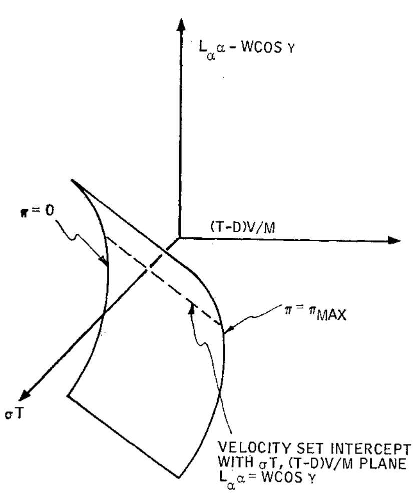

Fig. 7 Velocity set for equation set 3.

5. *Energy State Approximation*

In the energy state approximation, the equilibrium of vertical forces is assumed, and $\sin\alpha \cong \alpha$ and $\gamma = 0$. The energy state equations are:

$$\begin{aligned} \dot{E} &= (T - D)V/M \\ \dot{x} &= V \end{aligned} \tag{A15}$$

Here V and π are regarded as the control variables. The Hamiltonian for minimum fuel is

$$H = \sigma T + \lambda_1(T - D)V/M + \lambda_2 V \tag{A16}$$

The first and second partial derivatives are:

$$\begin{gathered} \partial H/\partial \pi = \sigma + \lambda_1 V/M = 0, \qquad \partial H/\partial V = 0 \\ [\partial^2 H/\partial u^2] = -(\lambda_1/M)^2 \end{gathered} \tag{A17}$$

But the second partial is not positive semidefinite; thus partial throttle control is not possible.

The velocity set is not convex for the assumed form of the data (parabolic drag polar and specific fuel consumption independent of power setting); thus a relaxed controller may exist and the maximum principle may not apply. If σ is assumed to vary with π and V, additional terms appear in $(\partial^2 H)/\partial u^2$ which could make it ≥ 0 and the velocity set convex.

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